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THICKNESS-TWIST VIBRATIONS OF A QUARTZ STRIP

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Abstract—An exact solution of the three-dimensional equations of elasticity has been found for the thickness-twist modes of vibration of a rotated-Y-cut quartz plate with a pair of parallel, free edges. The solution has a simple form: made possible by relaxing the condition that the edge-planes be perpendicular to the main faces of the plate. For the AT-cut, the edges are off perpendicular by about five degrees.

INTRODUCTION

IN A previous paper [1], the solution was given for the thickness-twist modes of vibration of infinite, rotated-Y-cut, quartz crystal plates. In the present paper, the solution is recast in a different form [2] and extended to accommodate similar plates of finite width.

A rotated-Y-cut plate is one whose middle plane contains a digonal axis of elastic symmetry of the quartz crystal and whose normal makes an angle (35° 15′ for the widely used AT-cut) with the trigonal axis [3]. In an isotropic plate, the thickness-twist modes of vibration are those in which the displacement direction is parallel to the middle plane of the plate and the wave normal is perpendicular to the displacement. In infinite, rotated-Y-cut quartz plates, analogous modes are possible [1] if the displacement direction is parallel to the digonal axis in the plane of the plate. Again in the isotropic plate, simple reflections of thickness-twist waves occur at free, plane boundaries of the plate parallel to the displacement direction and perpendicular to the middle plane. The same is not true in the quartz plate: with the result that the thickness-twist vibrations of a quartz strip, with a rectangular cross-section perpendicular to the digonal axis, are not expressible in terms of a finite number of elementary functions. The mode shape, if it were found, would comprise an infinite number of elementary shapes; and those with the longer wave lengths would be designated as "unwanted modes" or "spurious modes" which cause difficulties in crystal filter applications in electric circuits.

In the present paper, it is shown that, if the free, plane boundaries of the strip are maintained parallel to the digonal axis but cut slightly off normal to the middle plane (about 5° off normal for the AT-cut) simple reflections occur, a simple, closed solution is obtained and modes with simple shapes and well spaced frequencies result.

SOLUTION

If the x-axis is a digonal axis and the y-axis is normal to the middle plane of the plate, the stress–strain relations for the rotated-Y-cuts of quartz are [3]

$$\begin{aligned}
T_1 &= c_{11}S_1+c_{12}S_2+c_{13}S_3+c_{14}S_4, & T_2 &= c_{21}S_1+c_{22}S_2+c_{23}S_3+c_{24}S_4,\\
T_3 &= c_{31}S_1+c_{32}S_2+c_{33}S_3+c_{34}S_4, & T_4 &= c_{41}S_1+c_{42}S_2+c_{43}S_3+c_{44}S_4,\\
T_5 &= c_{55}S_5+c_{56}S_6, & T_6 &= c_{65}S_5+c_{66}S_6,
\end{aligned} \tag{1}$$

where $c_{pq} = c_{qp}$ and the strains, S_p, are related to the displacements u, v, w, according to

$$S_1 = \frac{\partial u}{\partial x}, \qquad S_4 = \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z},$$
$$S_2 = \frac{\partial v}{\partial y}, \qquad S_5 = \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}, \tag{2}$$
$$S_3 = \frac{\partial w}{\partial z}, \qquad S_6 = \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}.$$

The stresses must satisfy the equations of motion:

$$\frac{\partial T_1}{\partial x}+\frac{\partial T_6}{\partial y}+\frac{\partial T_5}{\partial z} = \rho\frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial T_6}{\partial x}+\frac{\partial T_2}{\partial y}+\frac{\partial T_4}{\partial z} = \rho\frac{\partial^2 v}{\partial t^2}, \qquad \frac{\partial T_5}{\partial x}+\frac{\partial T_4}{\partial y}+\frac{\partial T_3}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2}. \tag{3}$$

We require a steady state solution, of these equations, of the form

$$u = U(y, z)\,e^{i\omega t}, \qquad v = w = 0, \tag{4}$$

for a strip of finite thickness in the y-direction and of finite width in the z-direction—with all four boundaries free of traction.

Substitution of (4) in (1)–(3) reduces the latter to

$$T_1 = T_2 = T_3 = T_4 = 0, \qquad T_5 = c_{55}\frac{\partial U}{\partial z}+c_{56}\frac{\partial U}{\partial y}, \qquad T_6 = c_{65}\frac{\partial U}{\partial z}+c_{66}\frac{\partial U}{\partial y}; \tag{5}$$

$$S_1 = S_2 = S_3 = S_4 = 0, \qquad S_5 = \frac{\partial U}{\partial z}, \qquad S_6 = \frac{\partial U}{\partial y}; \tag{6}$$

$$c_{66}\frac{\partial^2 U}{\partial y^2}+2c_{56}\frac{\partial^2 U}{\partial y\partial z}+c_{55}\frac{\partial^2 U}{\partial z^2} = -\rho\omega^2 U; \tag{7}$$

with the exponential factor omitted.

Now, the four functions [2]

$$\begin{aligned} U = {} & A\sin\eta y\cos\zeta\left(\frac{c_{56}}{c_{66}}y-z\right)+B\sin\eta y\sin\zeta\left(\frac{c_{56}}{c_{66}}y-z\right) \\ & +C\cos\eta y\cos\zeta\left(\frac{c_{56}}{c_{66}}y-z\right)+D\cos\eta y\sin\zeta\left(\frac{c_{56}}{c_{66}}y-z\right) \end{aligned} \tag{8}$$

are solutions of (7) if

$$\rho\omega^2 = c_{66}\eta^2+\gamma_{55}\zeta^2, \tag{9}$$

where

$$\gamma_{55} = c_{55}-c_{56}^2/c_{66}. \tag{10}$$

For a plate of finite thickness, say $2h$, we require

$$T_2]_{y=\pm h} = T_4]_{y=\pm h} = T_6]_{y=\pm h} = 0; \tag{11}$$

and these boundary conditions are satisfied by (8) if

$$2\eta h = m\pi, \tag{12}$$

where m is an odd integer for solutions A and B and an even integer (including zero—which corresponds to a face-shear mode) for solutions C and D.

For traction-free boundaries at $z = \pm c$, we should require

$$T_3]_{z=\pm c} = T_4]_{z=\pm c} = T_5]_{z=\pm c} = 0. \tag{13}$$

The first two of these are satisfied identically, but the third cannot be satisfied by functions as simple as (8). An infinite series of such functions would be required. However, consider a rotation of coordinate axes, through an angle α about the x-axis, according to the scheme of direction cosines

$$\begin{array}{cccc} & x & y & z \\ x' & 1 & 0 & 0 \\ y' & 0 & \cos\alpha & \sin\alpha \\ z' & 0 & -\sin\alpha & \cos\alpha \end{array} \tag{14}$$

With (5), the components of stress, referred to the rotated axes, are given in terms of the original components by

$$T'_1 = T'_2 = T'_3 = T'_4 = 0, \qquad T'_5 = T_5\cos\alpha - T_6\sin\alpha, \qquad T'_6 = T_5\sin\alpha + T_6\cos\alpha; \tag{15}$$

and, for traction-free boundaries at $z' = \pm c\cos\alpha$, as shown in Fig. 1, we require

$$T'_3]_{z'=\pm c\cos\alpha} = T'_4]_{z'=\pm c\cos\alpha} = T'_5]_{z'=\pm c\cos\alpha} = 0. \tag{16}$$

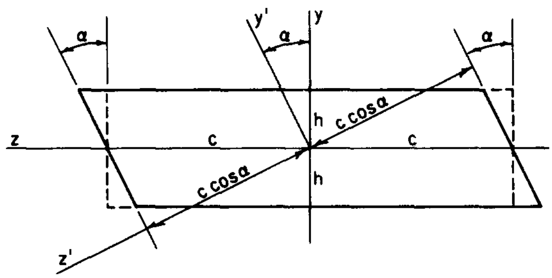

FIG. 1. Rotated axes and cross-section of quartz strip.

Now, T'_3 and T'_4 are identically zero whereas

$$\begin{aligned} T'_5 = {} & A\eta(c_{56}\cos\alpha - c_{66}\sin\alpha)\cos\eta y\cos\zeta\left(\frac{c_{56}}{c_{66}}y - z\right) + A\zeta\gamma_{55}\cos\alpha\sin\eta y\sin\zeta\left(\frac{c_{56}}{c_{66}}y - z\right) \\ & + B\eta(c_{56}\cos\alpha - c_{66}\sin\alpha)\cos\eta y\sin\zeta\left(\frac{c_{56}}{c_{66}}y - z\right) - B\zeta\gamma_{55}\cos\alpha\sin\eta y\cos\zeta\left(\frac{c_{56}}{c_{66}}y - z\right) \\ & - C\eta(c_{56}\cos\alpha - c_{66}\sin\alpha)\sin\eta y\cos\zeta\left(\frac{c_{56}}{c_{66}}y - z\right) + C\zeta\gamma_{55}\cos\alpha\cos\eta y\sin\zeta\left(\frac{c_{56}}{c_{66}}y - z\right) \\ & - D\eta(c_{56}\cos\alpha - c_{66}\sin\alpha)\sin\eta y\sin\zeta\left(\frac{c_{56}}{c_{66}}y - z\right) - D\zeta\gamma_{55}\cos\alpha\cos\eta y\cos\zeta\left(\frac{c_{56}}{c_{66}}y - z\right) \end{aligned} \tag{17}$$

In (17), set

$$\alpha = \arctan(c_{56}/c_{66}) \tag{18}$$

and note that $(c_{56}/c_{66})y - z = -z'/\cos\alpha$. Then (17) reduces to

$$\begin{aligned} T'_5 &= A\zeta\gamma_{55}\cos\alpha\sin\eta y\sin(\zeta z'/\cos\alpha) - B\zeta\gamma_{55}\cos\alpha\sin\eta y\cos(\zeta z'/\cos\alpha) \\ &\quad + C\zeta\gamma_{55}\cos\alpha\cos\eta y\sin(\zeta z'/\cos\alpha) - D\zeta\gamma_{55}\cos\alpha\cos\eta y\cos(\zeta z'/\cos\alpha). \end{aligned} \tag{19}$$

Hence, it is apparent that $T'_5 = 0$ on $z' = \pm c\cos\alpha$ if

$$2\zeta c = n\pi, \tag{20}$$

where n is an even integer (including zero—which gives simple thickness-shear modes) for solutions A and C, and an odd integer for solutions B and D.

Upon substituting (12) and (20) in (8) and (9), we find the required simple solutions:

$$\begin{aligned} U &= A\sin\frac{m\pi y}{2h}\cos\frac{n\pi}{2c}\left(\frac{c_{56}}{c_{66}}y - z\right) + B\sin\frac{m\pi y}{2h}\sin\frac{n\pi}{2c}\left(\frac{c_{56}}{c_{66}}y - z\right) \\ &\quad + C\cos\frac{m\pi y}{2h}\cos\frac{n\pi}{2c}\left(\frac{c_{56}}{c_{66}}y - z\right) + D\cos\frac{m\pi y}{2h}\sin\frac{n\pi}{2c}\left(\frac{c_{56}}{c_{66}}y - z\right). \end{aligned} \tag{21}$$

with circular frequencies

$$\omega = \frac{m\pi}{2h}\left(\frac{c_{66}}{\rho}\right)^{\frac{1}{2}}\left(1 + \frac{\gamma_{55}n^2h^2}{c_{66}m^2c^2}\right)^{\frac{1}{2}} \tag{22}$$

and m and n odd or even integers, for solutions A, B, C, D, according to

A: m odd, n even;

B: m odd, n odd;

C: m even, n even;

D: m even, n odd.

For the AT-cut of quartz $c_{56} = 2{\cdot}53$ and $c_{66} = 29{\cdot}01$, in units of 10^{10} dyn/cm^2, as calculated from Bechmann's values [4] of the principal constants of quartz. Hence, by (18), α is approximately 5° for the AT-cut. Thus, only a slight inclination of the edge-cuts is required to reduce the undesirable, complicated mode-shapes to simple ones.

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Абстракт—Приводится точное решение для трехмерных уравнений упругости для случая крутильных по толщине видов колебаний вращающейся пластинки из крарца Y–среза с парой параллельных, свободных краев.

Решение имеет простую форму и даёт возможность ослабить условия так, что краевые плоскости могут быть перпедикулярными к главным поверхностям пластинки. Для кварца AT–среза, края отдаляются от перпедикулятности около пяти степеней.